On a p-adic interpolation function for the multiple generalized Euler numbers and its derivatives

Taekyun Kim

Division of General Education-Mathematics,
Kwangwoon University, Seoul 139-701, Korea
e-mail: tkkim@kw.ac.kr

Abstract : We study analytic function interpolating the multiple generalized Euler numbers attached to χ at negative integers in complex plane and we consider the multiple p-adic l-function as the p-adic analog of the above function. Finally, we give the value of the partial derivative of this multiple p-adic l-function at $s = 0$.

2000 Mathematics Subject Classification : 11B68, 11S40, 11S80

Key words : Euler numbers, Euler polynomials, multiple generalized Euler numbers, multiple generalized Euler numbers attached to χ, multiple p-adic l-function

1. Introduction

Let p be a fixed odd prime number. Throughout this paper $\mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}$ and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integers, the field of p-adic rational numbers, the complex number field and the completion of algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. The Euler numbers in $\mathbb{C}$ are defined by the formula

$$F(t) = \frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!} \text{ for } |t| < \pi, \text{ see [1-6].} \tag{1}$$

It follows from the definition that $E_0 = 1, E_1 = -1/2, E_2 = 0, E_3 = 1/4, \cdots$, and $E_{2k} = 0$ for $k = 1, 2, 3, \cdots$.

Let $r \in \mathbb{N}$. Then the multiple Euler numbers of order r are defined as

$$F^{(r)}(t) = \left(\frac{2}{e^t + 1}\right)^r = \sum_{n=0}^{\infty} E_n^{(r)} \frac{t^n}{n!}, |t| < \pi.$$

Let x be a variable. Then the multiple Euler polynomials are also defined by the rule

$$F^{(r)}(t, x) = \left(\frac{2}{e^t + 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x) \frac{t^n}{n!}. \tag{2}$$

For $f \in \mathbb{N}$ with $f \equiv 1 \pmod 2$, assume that χ is a primitive Dirichlet character with conductor f. It is known that the generalized Euler numbers attached to χ, E_n, χ, are defined by the rule

$$F_\chi(t) = 2 \sum_{a=1}^{f} \frac{\chi(a)(-1)^a e^{at}}{e^{ft} + 1} = \sum_{n=0}^{\infty} E_{n,\chi} \frac{t^n}{n!}, \tag{3}$$

where $|t| < \frac{\pi}{f}$, (see [1]).

In this paper, we consider the multiple generalized Euler numbers attached to χ in the sense of the multiple of Eq.(3). From these numbers, we study an analytic function interpolating the multiple generalized Euler numbers attached to χ at negative integers in complex number field. In the sense of p-adic analog of the above function and give the values of the partial derivative of this multiple p-adic l-function at $s = 0$.

2. Analytic functions associated with Euler numbers and polynomials

We now consider the multiple generalized Euler numbers attached to χ, $E_{n,\chi}^{(r)}$, which are defined by

$$F_{\chi}^{(r)}(t) = 2^r \sum_{a_1,\cdots,a_r=1}^{f} \frac{(-1)^{\sum_{i=1}^r a_i}\chi(a_1+\cdots+a_r)e^{t\sum_{i=1}^r a_i}}{(e^{ft}+1)^r} = \sum_{n=0}^{\infty} E_{n,\chi}^{(r)}\frac{t^n}{n!}. \tag{4}$$

By (2) and (4), we readily see that

$$E_{n,\chi}^{(r)} = f^n \sum_{a_1,\cdots,a_r=0}^{f-1} (-1)^{\sum_{i=1}^r a_i}\chi(a_1+\cdots+a_r)E_n^{(r)}\left(\frac{a_1+\cdots+a_r}{f}\right). \tag{5}$$

For $s \in \mathbb{C}$, we have

$$2^r \sum_{n_1,\cdots,n_r=0}^{\infty} \frac{(-1)^{n_1+\cdots+n_r}}{(x+n_1+\cdots+n_r)^s} = \frac{1}{\Gamma(s)}\int_0^{\infty} F^{(r)}(-t,x)t^{s-1}dt, \tag{6}$$

where $x \neq 0, -1, -2, \cdots$. From (6), we can consider the multiple Euler zeta function as follows:

$$\zeta_r(s,x) = 2^r \sum_{n_1,\cdots,n_r=0}^{\infty} \frac{(-1)^{n_1+\cdots+n_r}}{(x+n_1+\cdots+n_r)^s}, \text{ for } x \in \mathbb{C}, x \neq 0, -1, -2, \cdots. \tag{7}$$

By (1), (2), (5), (6) and (7), we easily see that

$$\zeta_r(-n,x) = E_n^{(r)}(x) \text{ for } n \in \mathbb{N}.$$

By using complex integral and (4), we can also obtain the following equation: for $s \in \mathbb{C}$,

$$\frac{1}{\Gamma(s)}\int_0^{\infty} F_{\chi}^{(r)}(-t)t^{s-1}dt = 2^r \sum_{\substack{n_1,\cdots,n_r=0 \\ n_1+\cdots+n_r\neq 0}}^{\infty} \frac{\chi(n_1+\cdots+n_r)(-1)^{n_1+\cdots+n_r}}{(n_1+\cdots+n_r)^s}, \tag{8}$$

where χ is the primitive Dirichlet character with conductor $f \in \mathbb{N}$ with $f \equiv 1 \pmod 2$.

By (8), we define Dirichlet's type multiple Euler l-function in complex plane as follows: for $s \in \mathbb{C}$,

$$l_r(s,\chi) = 2^r \sum_{\substack{n_1,\cdots,n_r=0 \\ n_1+\cdots+n_r\neq 0}}^{\infty} \frac{\chi(n_1+\cdots+n_r)(-1)^{n_1+\cdots+n_r}}{(n_1+\cdots+n_r)^s}. \tag{9}$$

From (4), (8) and (9), we note that

$$l_r(-n, \chi) = E_{n,\chi}^{(r)} \text{ for } n \in \mathbb{N}. \tag{10}$$

Let s be a complex variable, and let a and F be integer with $0 < a < F$ and $F \equiv 1 \pmod 2$. Then we consider partial zeta function $T_r(s; a_1, \cdots, a_r|F)$ as follows:

$$\begin{aligned} T_r(s; a_1, \cdots, a_r|F) &= 2^r \sum_{\substack{m_1,\cdots,m_r>0 \\ m_i \equiv a_i \pmod F}} \frac{(-1)^{m_1+\cdots+m_r}}{(m_1+\cdots+m_r)^s} \\ &= (-1)^{a_1+\cdots+a_r} F^{-s} \zeta_r\left(s, \frac{a_1+\cdots+a_r}{F}\right). \end{aligned} \tag{11}$$

Let $\chi(\neq 1)$ be the Dirichlet character with conductor $f \in \mathbb{N}$ with $f \equiv 1 \pmod 2$ and let F be the multiple of f with $F \equiv 1 \pmod 2$. Then Dirichlet's type multiple l-function can be expressed as the sum

$$l_r(s, \chi) = \sum_{a_1,\cdots,a_r=1}^{F} \chi(a_1+\cdots+a_r) T_r(s; a_1, \cdots, a_r|F) \text{ for } s \in \mathbb{C}. \tag{12}$$

By simple calculation, we easily see that

$$T_r(-n; a_1, \cdots, a_r|F) = F^n (-1)^{a_1+\cdots+a_r} E_n^{(r)}\left(\frac{a_1+\cdots+a_r}{F}\right) \text{ for } r, n \in \mathbb{N}. \tag{13}$$

From (5), (12) and (13), we note that

$$l_r(-n, \chi) = E_{n,\chi}^{(r)} \text{ for } n \in \mathbb{N}. \tag{14}$$

By (1) and (2), we have

$$E_n^{(r)}(x) = \sum_{l=0}^{n} \binom{n}{l} E_l^{(r)} x^{n-l} = \sum_{l=0}^{n} \binom{n}{l} E_{n-l}^{(r)} x^l. \tag{15}$$

From (11), (13) and (15), we derive the following equation:

$$T_r(s; a_1, \cdots, a_r|F) = (-1)^{a_1+\cdots+a_r} (a_1+\cdots+a_r)^{-s} \sum_{k=0}^{\infty} \binom{-s}{k} \left(\frac{F}{a_1+\cdots+a_r}\right)^k E_k^{(r)}. \tag{16}$$

It follows from (12), (13) and (16) that

$$\begin{aligned} l_r(s, \chi) = &\sum_{a_1,\cdots,a_r=1}^{F} \chi(a_1+\cdots+a_r)(-1)^{a_1+\cdots+a_r}(a_1+\cdots+a_r)^{-s} \\ &\times \sum_{m=0}^{\infty} \binom{-s}{m} \left(\frac{F}{a_1+\cdots+a_r}\right)^m E_m^{(r)}. \end{aligned} \tag{17}$$

The values of $l_r(s, \chi)$ at negative integer are algebraic numbers, and hence they can be regarded as numbers belonging to an extension of $\mathbb{Q}_p$. In the next section, we therefore look for a p-adic function that agrees with $l_r(s, \chi)$ at negative integers.

3. On a p-adic interpolation function for the multiple Euler numbers and its derivative

In this section, we study the p-adic analogs of the Dirichlet's type multiple Euler l-function, $l_r(s,\chi)$, which were introduced in previous section. If fact, this function is the p-adic interpolation function for the multiple generalized Euler numbers attached to χ at negative integers. Let w denote the Teichmüller character with the conductor $f_w = p$. For an arbitrary character χ, we set $\chi_n = \chi w^{-n}, n \in \mathbb{Z}$, in the sense of the product of characters. Let

$$< a >= w^{-1}(a)a = \frac{a}{w(a)}.$$

Then we note that $< a >\equiv 1 \pmod{p\mathbb{Z}_p}$. Let

$$A_j(x) = \sum_{n=0}^{\infty} a_{n,j}x^n, a_{n,j} \in \mathbb{C}_p, j = 0, 1, 2, \cdots$$

be a sequence of power series, each convergent on a fixed subset

$$D = \{s \in \mathbb{C}_p \mid |s|_p < p^{1-\frac{1}{p-1}}\},$$

of $\mathbb{C}_p$ such that

(1) $a_{n,j} \to a_{n,0}$ as $j \to \infty$ for any n;

(2) for each $s \in D$ and $\epsilon > 0$, there exists an $n_0 = n_0(s, \epsilon)$ such that

$$|\sum_{n\geq n_0} a_{n,j}s^n|_p < \epsilon \text{ for } \forall j.$$

In this case,

$$\lim_{j\to\infty} A_j(s) = A_0(s), \text{ for all } s \in D.$$

This was used by Washington ([6]) to show that each of functions $w^{-s}(a)a^s$ and

$$\sum_{m=0}^{\infty} \binom{s}{m} \left(\frac{F}{a}\right)^m B_m,$$

where F is multiple of p and f and B_m is the m-th Bernoulli numbers, is analytic on D (see [6]).

Let χ be a primitive Dirichlet's character with conductor $f \in \mathbb{N}$ with $f \equiv 1 \pmod 2$. Then we consider the multiple Euler p-adic l-function, $l_{p,r}(s,\chi)$, which interpolates the multiple generalized Euler numbers attached to χ at negative integers.

For $f \in \mathbb{N}$ with $f \equiv 1 \pmod 2$, let us assume that F is a positive integral multiple of p and $f = f_\chi$. We now define the multiple Euler p-adic l-function as follows:

$$\begin{aligned} l_{p,r}(s,\chi) = &\sum_{a_1,\cdots,a_r=1}^{F} \chi(a_1+\cdots+a_r) < a_1+\cdots+a_r >^{-s} (-1)^{a_1+\cdots+a_r} \\ &\times \sum_{m=0}^{\infty} \binom{-s}{m} \left(\frac{F}{a_1+\cdots+a_r}\right)^m E_m^{(r)}. \end{aligned} \tag{18}$$

From (18), we note that $l_{p,r}(s,\chi)$ is analytic for $s \in D$.

For $n \in \mathbb{N}$, we have

$$E_{n,\chi_n}^{(r)} = F^n \sum_{a_1,\cdots,a_r=0}^{F-1} E_n^{(r)}\left(\frac{a_1+\cdots+a_r}{F}\right)\chi_n(a_1+\cdots+a_r)(-1)^{a_1+\cdots+a_r}. \tag{19}$$

If $\chi_n(p) \neq 0$, then $(p, f_{\chi_n}) = 1$, and thus the ratio F/p is a multiple of f_{χ_n}.

Let

$$I_0 = \left\{\frac{a_1+\cdots+a_r}{p} \mid a_1+\cdots+a_r \equiv 0 \pmod{p} \text{ for some } a_i \in \mathbb{Z} \text{ with } 0 \leq a_i \leq F\right\}.$$

Then we have

$$\begin{aligned} &F^n \sum_{\substack{a_1,\cdots,a_r=0 \\ p\mid a_1+\cdots+a_r}}^{F-1} E_n^{(r)}\left(\frac{a_1+\cdots+a_r}{F}\right)\chi_n(a_1+\cdots+a_r)(-1)^{a_1+\cdots+a_r} \\ &= p^n\left(\frac{F}{p}\right)^n \chi_n(p) \sum_{\substack{a_1,\cdots,a_r=0 \\ \beta\in I_0}}^{F/p} \chi_n(\beta)(-1)^{\beta} E_n^{(r)}\left(\frac{\beta}{F/p}\right). \end{aligned} \tag{20}$$

From (20), we define the second multiple generalized Euler numbers attached to χ as follows:

$$E_{n,\chi_n}^{*(r)} = \left(\frac{F}{p}\right)^n \sum_{\substack{a_1,\cdots,a_r=0 \\ \beta\in I_0}}^{F/p} \chi_n(\beta)(-1)^{\beta} E_n^{(r)}\left(\frac{\beta}{F/p}\right). \tag{21}$$

By (19), (20) and (21), we easily see that

$$\begin{aligned} &E_{n,\chi_n}^{(r)} - p^n\chi_n(p)E_{n,\chi_n}^{*(r)} \\ &= F^n \sum_{\substack{a_1,\cdots,a_r=1 \\ p\nmid a_1+\cdots+a_r}}^{F} \chi_n(a_1+\cdots+a_r)(-1)^{a_1+\cdots+a_r} E_n^{(r)}\left(\frac{a_1+\cdots+a_r}{F}\right). \end{aligned} \tag{22}$$

By the definition of the multiple Euler polynomials of order r, we see that

$$E_n^{(r)}\left(\frac{a_1+\cdots+a_r}{F}\right) = F^{-n}(a_1+\cdots+a_r)^n \sum_{k=0}^{n}\binom{n}{k}\left(\frac{F}{a_1+\cdots+a_r}\right)^k E_k^{(r)}. \tag{23}$$

From (22) and (23), we have

$$\begin{aligned} E_{n,\chi_n}^{(r)} - p^n\chi_n(p)E_{n,\chi_n}^{*(r)} = &\sum_{\substack{a_1,\cdots,a_r=1 \\ p\nmid a_1+\cdots+a_r}}^{F} (a_1+\cdots+a_r)^n\chi_n(a_1+\cdots+a_r)(-1)^{a_1+\cdots+a_r} \\ &\times \sum_{k=0}^{n}\binom{n}{k}\left(\frac{F}{a_1+\cdots+a_r}\right)^k E_k^{(r)}. \end{aligned} \tag{24}$$

By (18) and (24), we readily see that

$$\begin{aligned}
& l_{p,r}(-n,\chi) \\
&= \sum_{\substack{a_1,\cdots,a_r=1 \\ p\nmid a_1+\cdots+a_r}}^{F} (a_1+\cdots+a_r)^n \chi_n(a_1+\cdots+a_r)(-1)^{a_1+\cdots+a_r} \\
&\qquad \times \sum_{m=0}^{n} \binom{n}{m} \left(\frac{F}{a_1+\cdots+a_r}\right)^m E_m^{(r)} \\
&= E_{n,\chi_n}^{(r)} - p^n \chi_n(p) E_{n,\chi_n}^{*(r)}.
\end{aligned} \tag{25}$$

Therefore, we obtain the following theorem.

Theorem 1. Let F be a positive integral of p and $f(=f_{\chi_n})$, and let

$$\begin{aligned}
l_{p,r}(s,\chi) = & \sum_{a_1,\cdots,a_r=1}^{F} \chi(a_1+\cdots+a_r) < a_1+\cdots+a_r >^{-s} (-1)^{a_1+\cdots+a_r} \\
& \times \sum_{m=0}^{\infty} \binom{-s}{m} \left(\frac{F}{a_1+\cdots+a_r}\right)^m E_m^{(r)}.
\end{aligned}$$

Then $l_{p,r}(s,\chi)$ is analytic on D. Furthermore, for each $n \in \mathbb{N}$, we have

$$l_{p,r}(-n,\chi) = E_{n,\chi_n}^{(r)} - p^n \chi_n(p) E_{n,\chi_n}^{*(r)}.$$

Using Taylor expansion at $s=0$, we get

$$\binom{-s}{m} = \frac{(-1)^m}{m} s + \cdots \ \text{ if } m \geq 1. \tag{26}$$

From (26) and Theorem 1, we obtain the following corollary.

Corollary 2. Let F be a positive integral multiple of p and f. Then we have

$$\begin{aligned}
\frac{\partial}{\partial s} l_{p,r}(0,\chi) = & \sum_{\substack{a_1,\cdots,a_r=1 \\ (a_1+\cdots+a_r,p)=1}}^{F} \chi(a_1+\cdots+a_r)(-1)^{a_1+\cdots+a_r} \left(1-\log_p(a_1+\cdots+a_r)\right) \\
& + \sum_{\substack{a_1,\cdots,a_r=1 \\ (a_1+\cdots+a_r,p)=1}}^{F} \chi(a_1+\cdots+a_r)(-1)^{a_1+\cdots+a_r} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \left(\frac{F}{a_1+\cdots+a_r}\right)^m E_m^{(r)},
\end{aligned}$$

where $\log_p x$ is denoted by the p-adic logarithm.

REFERENCES

1. T. Kim, Barnes type multiple q-zeta function and q-Euler polynomials, J. Phys. A: Math. Theor., 43(2010) 255201(11pp).

2. T. Kim, q-Volkenborn integration, Russ. J. Math. phys., 9(2002), 288-299.

3. T. Kim, Note on the Euler q-zeta functions, J. Number Theory, 129(2009), 1798-1804.

4. H. Ozden, Y. Simsek, S.-H. Rim, I. N. Cancul, A note on p-adic q-Euler measure, Adv. Stud. Contemp. Math., 14(2007), 233-239.

5. K. Shiratani, S. Yamamoto On a p-adic interpolation function for the Euler numbers and its derivatives, Mem. Fac. Sci. Kyushu Univ. A, 39(1985), 113-125.

6. L. C. Washington, Introduction to Cyclotomic Field, Springer, 1982.